II. *An Account of a Book, intitled,* P. D. Pauli Frisii Mediolanensis, *&c.* Disquisitio mathematica in causam physicam figuræ et magnitudinis Telluris nostræ; *printed at* Milan *in* 1752. *inscribed to the Count* de Sylva, *and consisting of Ten Sheets and a half in Quarto: By Mr.* J. Short, *F. R. S.*

Read Jan. 18, 1753.

1. IT may be laid down as a rule in mix'd mathematics, "That the determination of no physical quantity be carried farther than the observations, or other mechanical measures, can bear;" lest there follow this incongruity, of the conclusion being more extensive than the premises. It were absurd, for instance, in the resolution of a triangle, to compute an angle to the exactness of seconds, or a side to centesms of an inch, when, perhaps, the instruments used can measure no angle less than 10 minutes, or a side but to the exactness of a foot The conclusions of arithmetic and geometry are indeed rigorously true, but they are only hypothetical; and whenever the quantities, that enter any practical question, cannot be measured, but within certain limits, it were in vain to look for an answer perfectly accurate. The error of the instrument becomes itself one of the *data*; and we must content ourselves to find the limits, which the quantity sought cannot well exceed, or fall short of, by such rules, as the great Mr. Cotes has left us in his excellent treatise on the subject.

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2. In like manner, when any phyfical theory is deduced from obfervations, its accuracy will ftill be in proportion to that of the obfervations, on which it is founded. Sir Ifaac Newton, we find, in computing the ratio of the earth's axis to its equatorial diameter, confines himfelf to a reafonable approximation, and to three places of figures (229 to 230); becaufe, whether that ratio is deduced from the different lengths of ifochronous pendulums in different latitudes, from the meafurement of diftant degrees of a meridian, or from both; the elements of the calculus can fcarcely furnifh a greater degree of exactnefs. And of the fame judicious caution we have many other examples in the works of that incomparable author.

3. On the other hand, when obfervations and theories are brought together, and compared, nothing can be juftly inferred againft a theory from its difagreement with the obfervations, unlefs that difagreement is greater, than can be fairly imputed to the imperfection of inftruments, and to the unavoidable miftakes of an obferver; efpecially, if the difference fhould be fometimes in excefs, and at other times in defect; or, according to fome of the obfervations, fhould intirely vanifh.

4. Altho' thefe rules, manifeftly well-founded, have been followed by all the beft writers, our author obferves, that feveral ingenious men, both in France, and in Italy, have deviated from them; particularly in treating of the famous queftion concerning the figure of the earth. Some, with Meffieurs Clairaut and Bouguer, attributing too much to the obfervations, that have been made, and taking them

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for abſolutely exact, have concluded Sir Iſaac Newton's reaſonings on that ſubject to be faulty; while Father Boſcowic, a Jeſuit at Rome, making them quite looſe and uncertain, thinks no argument at all can be drawn from them, concerning the earth's figure; far leſs in confirmation of the Newtonian theory.

5. In oppoſition to theſe two extremes, equally contrary to reaſon, as they are to each other, F. Friſi writes the treatiſe now before us; in the introduction to which he ſhews, 1. That, altho' the ratio of the axis of the earth to its equatorial diameter is, from M. de Maupertuis's operations in Lapland, and afterwards in France, that of 177 to 178; and by the theory only 229 to 230; yet the difference is no more, than what might ariſe from a miſtake of about 60 toiſes in the meaſure of either of the two degrees, that are compared, or of 30 toiſes in each of them. Or, ſuppoſe the meaſure of the arcs to be exact, the ſame difference might be owing to an error of 4 or 5 ſeconds in the aſtronomical part. And ſuch errors, or others equivalent to them, in a courſe of ſo many combined operations, our author looks upon as difficult to be avoided. But he adds, if the obſervations of M. de Maupertuis, and his fellow academicians, ſeem to differ from the theory, thoſe of Meſſieurs Bouguer and de la Condamine exactly agree with it: According to whom, a degree at the equator, containing 56753 toiſes, and in latitude 49° 22′ 57183 toiſes, the difference of the axis and equatorial diameter comes out to be $\frac{1}{229}$.

6. In anſwer to F. Boſcowic, and thoſe who make no account of the obſervations, our author allows, that,

that, if they were ſuch, as M. Caſſini, and ſome other academicians, made in France, of the meaſure of a parallel of latitude, they could not be much depended on; that method being liable to ſeveral obvious inconveniencies. But he inſiſts, that, with the excellent inſtruments, which were uſed, and, conſidering the diſtinguiſhed ſkill of the obſervers, as well at the polar circle as in France, and at the equator, the error upon one degree of the meridian could not exceed 60 or 70 toiſes; which is a degree of exactneſs not only ſufficient for the determination of the firſt queſtion, *viz.* whether the ſpheroid of the earth is flat or long; but likewiſe to found an agreement between the obſervations and the theory, as near as can be expected or deſired.

7. The work itſelf is divided into ten chapters:

I. De obſervationibus circa telluris figuram hactenus inſtitutis.
II. De principiis et hypotheſibus quibuſdam.
III. De rotatione corporum, et vi centrifuga.
IV. De mutationibus ex motu circulari ortis.
V. De attractione corporum rotundorum.
VI. De comparatione gravitatis in variis homogeneæ ſphæroidis locis.
VII. De figura terræ.
VIII. De gradibus meridiani et parallelorum.
IX. De loxodromiis nautarum, de parallaxi lunæ, et aliis ex eadem theoria pendentibus.
X. De theoriæ et obſervationum conſenſu.

8. In chap. 1. we have a ſhort hiſtory of the inquiries, that have been made into the magnitude and figure

figure of the earth down to the present times; and the preference is justly given to the measurements of Mr. Norwood in England, A. D. 1635, and of the members of the French Academy of Sciences since that time. From these he gathers, that, within less than 60 or 70 toises, the lengths of a degree of the meridian are as follows:

	°	′	Toises.
Lat.	0	0	56753
	45	0	57100
	49	22	57183
	53	0	57300
	66	20	57400

9. Chap. II. contains an account of the principles, on which this theory is founded; *viz.* the universal gravitation of matter, and the diurnal rotation of the earth. Our author mentions likewise the hypothesis of the earth's being originally in a fluid state; but rejects it as precarious and improbable. He allows however, that, with regard to the present question, it is all one whether it was first a fluid or not, seeing the ocean is circumfused just in the same manner, and to the same altitude, as if the whole was still a fluid.

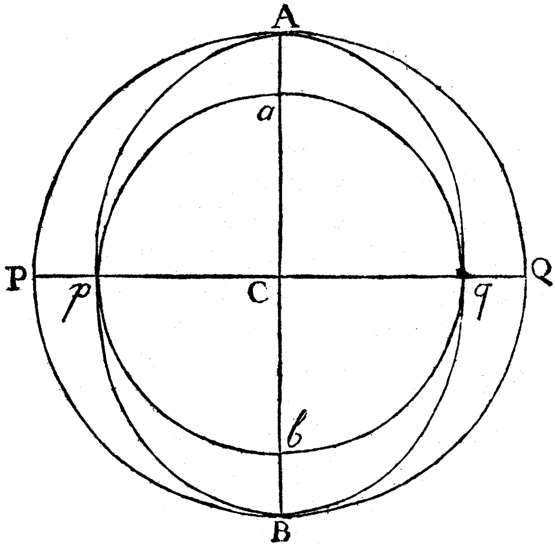
10. Chap. III. and IV. are employed in the doctrine of centrifugal forces, and their effect in changing a fluid sphere into the form of an oblate spheroid. In the former of these chapters, the author resolves, as usual, the centrifugal force of a particle into two others; one, that acts directly contrary to the gravitation of the particle; and the other a force in a direction perpendicular to it. And this last he considers

again as acting laterally upon the contiguous particles impelling them towards the equator. But the quantity of this force, when greatest at the octant, he computes to be only $\frac{1}{668168}$ of the force of gravity; and therefore, says he, it may be safely neglected. The truth is, after the spheroid is come to be in a permanent state, and all its parts in *æquilibrio*, there is no longer any such lateral force at all; it being now intirely satisfied by the gradual contraction of the earth's axis.

11. The general contents of the following chapters are sufficiently expressed in their titles already given. Nor can we be more particular, without entering into a detail of algebraical operations, which would be improper for this place; and which is the less necessary, as the same things have been treated of by several other authors.

This does not, however, in the least detract from the merit of F. Frisi; who discovers throughout this work much acuteness and skill, joined with all the candor and ingenuity, that become a philosopher. And as he has not yet exceeded his 23d year, it may be expected, that the sciences will one day be greatly indebted to him; especially as we find him actually engaged in composing a complete body of physico-mathematical learning.

12. There is only, in his VI. Chapter, a criticism upon one of Sir Isaac Newton's demonstrations, in which we cannot agree with this ingenious author. And as this demonstration has proved a stumbling-block, not only to F. Frisi, but to many other learned men, we shall be obliged to consider that part of it, which has been mistaken, at some length, by the help of the following scheme:



In which let the ellipsis *ApBqA*, whose axes *AB*, *pq*, are in any given ratio, as of *m* to *n*, have the circles *apbq*, and *APBQ*, inscribed and circumscribed to it: And if the figure revolves on the axis *PQ*, there will be generated an oblate spheroid *ApBqA*, with two spheres, the greater circumscribed to the spheroid, and touching it in its equator *APA*, and the lesser inscribed and touching it in the poles *pq*; the solid content of the spheroid being the first of the two mean proportionals between the solidity of the exterior sphere, and that of the interior.

But if the figure revolves on the axis *AB*, there will be generated a prolate spheroid *ApBqA*, inscribed in the exterior sphere at the poles *AB*; and circumscribing the interior sphere at the equator *pqp*, its solidity being the second of the above mean proportionals. So that if O and P ſtand for the ſolidities of the oblate and prolate ſpheroids, and S, s, for the two ſpheres; $S : O : P : s$ ∻ are in the continued proportion of $m : n$. And $S : P$, or $O : s :: m^2 : n^2$. As $S : s :: m^3 : n^3$.

Or we may with Sir Iſaac Newton conceive of the geneſis of theſe ſolids as follows. 1. Let the ſphere $APBQ$ be uniformly compreſſed in the direction of its axis PQ, till that axis is diminiſhed to pq, and the ſphere changed into the oblate ſpheroid. 2. Let this ſpheroid be equally compreſſed in the direction of that diameter of its equator, which is perpendicular to pq and AB, or to the plane of the figure; and it will degenerate into the prolate ſpheroid, whoſe poles are A and B. 3. Let this laſt be compreſſed in the direction of its axis AB, till it is changed into the ſphere $apbq$; and, in each of theſe compreſſions, the ſolid ſpace, which the body contains will be diminiſhed in the ratio of m to n.

Now, as the determination of the earth's figure depends not only upon that of the ratio of the centrifugal force, by which a body tends to recede from the axis of rotation to the power of gravity; but likewiſe, upon the decrement of gravitation ariſing from the body's being in that rotation actually removed to a greater diſtance from the centre: it is not enough, that we know, from the experiments with pendulums, the centrifugal force at the equator to be about $\frac{1}{289}$ of the force of gravity. We need, farther, two diſtinct propoſitions; one to determine the attractive force of a ſpheroid at its pole; and the other to determine its attraction at the equator. The firſt of theſe we have in *Princip. lib.* 1. *prop.* 91.

91. and the ſecond has been ſupplied by ſeveral authors. But Sir Iſaac, who ſeldom does any thing in vain, found, that he could, by one of his artifices, make that 91ſt propoſition ſerve likewiſe to determine the attraction at the equator, by the following argument:

Let G be the attraction of the exterior ſphere at A; and let the decrement of that attraction, when the ſphere is diminiſhed into the oblate ſpheroid $ApBq$, be d; and δ the decrement of this laſt attraction, when the oblate ſpheroid is diminiſhed into the prolate, whoſe poles are AB: then, I ſay, d is nearly equal to δ; the difference of the axes of the generating ellipſe being ſmall.

For the attractive matter, that is taken away, has, in both caſes, the ſame ratio to the matter, that is left; and its poſition, with reſpect to that which is left, is, in both caſes, nearly the ſame: And therefore the ſucceſſive attractions will be nearly in continued proportion, $G : G - d :: G - \overline{d + \delta}$ ÷. Or multiplying and rejecting d^2 as inconſiderable, $Gd = G\delta$, and $d = \delta$.

Thus, if the attractions of the ſphere $APBQ$, and of the prolate ſpheroid, at its pole A, be 126 and 125 reſpectively; the attraction of the intermediate oblate ſpheroid at its equator will be 125½: and how nearly this approaches to the truth, may be ſeen from an exact computation of thoſe attractions. For, if the axes of the generating ellipſe be 101 and 100, and the attractive force at the ſurface of the ſphere 126; the attraction at the pole of the prolate ſpheroid will be 124.9838; and that at the equator of the oblate 125.5077; which exceeds the arithmetical mean

between

between the two former, only by .0068; that is, by about $\frac{1}{18410}$th part of the attraction of the oblate spheroid at the equator.

This reasoning is more shortly expressed (*Princip. lib.* iii. *Prop.* 19.) as follows:

.... "Gravitas in loco *A* in sphæroidem, con-
"volutione ellipseos (*ApBq*) circa axem *AB* de-
"scriptam, est ad gravitatem in eodem loco *A* in
"sphæram centro *C* radio *AC* descriptam, ut 125
"ad 126. Est autem gravitas in loco *A* in terram
"media proportionalis inter gravitates in dictam
"sphæroidem et sphæram; propterea quod sphæra,
"diminuendo diametrum *PQ* in ratione 101 ad 100,
"vertitur in figuram terræ; et hæc figura, diminu-
"endo in eadem ratione diametrum tertiam, quæ dia-
"metris *AP*, *PQ* perpendicularis est, vertitur in dic-
"tam sphaeroidem; et gravitas in *A*, in *utroque casu*,
"diminuitur in *eadem ratione* quam proxime."

In which the expression "*eadem ratione*" occurring a second time has misled F. Frisi and others, to think this last ratio to be likewise that of the axes, or of 101 to 100: Whereas the *identity* of ratio's here asserted is to be referred only to the words "*utroque casu*;" the ratio itself being not that of the axes, or of *m* to *n*; but the half of that ratio (whatever it is found to be by *Prop.* 91. *lib.* i.) which the attraction of the sphere has to the polar attraction of the inscribed spheroid.

This inadvertence, however, of his own F. Frisi charges upon Sir Isaac Newton; and files it up, as the sixth of the errors, which he says have been discovered in the *Principia*. ... "Ita dum stabilitæ
"in 19 *lib.* iii. propositione terrestrium axium pro-
"portionis

"portionis fulcimentum et patrocinium quærimus, "aliud in propositione eadem *sophisma* sese offert, "quod eorum, quæ in principiis mathematicis New-"toni nacta (*i. e.* detecta) sunt hactenus, sextum "est, &c." But we may take it off the file again; and, for the present, leave the other five, till they are considered of at more leisure.

13. In his 10th and last chapter, our author sums up the evidence, and finds, that all the good observations, that have been made, as well by pendulums, as by actual mensuration, concur with the theory, in making the ratio of the earth's axis and equatorial diameter to be as 229 to 230. This is indeed a sufficient confirmation of the theory of gravitation: But it must be observed, that the coincidence is not perhaps quite so perfect as F. Frisi imagines. That ratio corresponds well enough to the exactness, to which the first elements of the calculus can be obtained; *the length of a second pendulum, and that of the earth's equatorial diameter*, from which the centrifugal force ($\frac{1}{289}$) is deduced. But, if we suppose that force to be accurately $\frac{1}{289}$, and compute more rigorously, we shall find the ratio in question to be very nearly that of 225 to 226; agreeing still with the observations as well as can be desired; and shewing, at the same time, the inimitable art of Sir Isaac Newton in the contrivance and use of approximations; seeing the strictest calculation raises the equator not the third part of a mean geographical mile above what he had found by his method.

I sent F. Frisi's book to my ingenious and learned friend the reverend Mr. Murdock, Fellow of this Society, who has fully consider'd the question concerning

cerning the figure of the earth; and who, after having peruſed the book, and diſcover'd the above miſtake of F. Friſi, ſent me the above theorem, and its demonſtration. He likewiſe ſent me the following theorems, which, he ſays, he had communicated to M. de Bremond, in the year 1740, when he was tranſlating his treatiſe on ſailing: But M. de Bremond dying ſoon after, thoſe, who had the care of publiſhing the tranſlation, printed it incorrect in ſeveral places; particularly the theorems for the prolate ſpheroid: On which account, he ſays, if they are thought worth preſerving, they may be inſerted in the *Philoſophical Tranſactions.*

Poſtſcript.

Theorems for computing the ratio of the attractive force of a ſpheroid, at its pole or equator, to that of the inſcribed or circumſcribed ſphere.

1. In an oblate ſpheroid, the ratio is,

$$\begin{matrix}\text{Pole} \\ \text{Equator}\end{matrix}\left\}\left\{\begin{matrix} 1+\dfrac{1}{m^2-1}-\dfrac{m^2}{\overline{m^2-1}|^{\frac{3}{2}}}\times A : \frac{1}{3} \\ \dfrac{m^2}{\overline{m^2-1}|^{\frac{3}{2}}}\times A-\dfrac{1}{m^2-1} : \frac{2}{3}\end{matrix}\right\}\right.$$

2. In a prolate ſpheroid, the ratio is,

$$\begin{matrix}\text{Pole} \\ \text{Equator}\end{matrix}\left\}\left\{\begin{matrix} 1-\dfrac{m^2}{m^2-1}+\dfrac{m}{\overline{m^2-1}|^{\frac{3}{2}}}\times l : \frac{1}{3} \\ \dfrac{m^2}{m^2-1}-\dfrac{m}{\overline{m^2-1}|^{\frac{3}{2}}}\times l : \frac{2}{3}\end{matrix}\right\}\right.$$

In

In which $m:1$, as the greater axis of the generating ellipſe is to the leſſer. A is a circular arc, to the radius 1, whoſe tangent is $\sqrt{m^2-1}$, or its reciprocal, if $m^2-1 = 1$. And l is the natural logarithm of $\frac{S}{V}$, S being the ſine of the arc, whoſe co-ſine is $\frac{1}{m} \times \sqrt{m^2-1}$, and V the verſed ſine of the ſame arc.

Note, The two firſt theorems, by ſubſtituting t for $\sqrt{m^2-1}$, coincide with thoſe of Mr. Maclaurin for the oblate ſpheroid, in his diſſertation on the tides.

III. *A Letter from the Rev. Mr.* George Coſtard, *Fellow of* Wadham-College, Oxford, *to Dr.* Bevis, *concerning the Year of the Eclipſe foretold by* Thales.

Dear Sir,

Read Jan. 25, 1753.

I THANK you greatly for the uſe of the Peterſburg Acts, while in London, where Bayer ſuppoſeth the eclipſe foretold by Thales to the Ionians fell out the year before Chriſt 603. Since my return home, upon looking over ſome papers, that I had formerly drawn up on that ſubject, I find, that I had determined it to have been the very ſame year. I will not trouble you with the chronological arguments, on which I founded that determination; and therefore ſhall only tranſcribe ſo much of thoſe papers as relates to calculation.

 Riccioli